ISRAEL JOURNAL OF MATHEMATICS, Vol. 33, Nos. 3–4, 1979

EVENTUALLY INDEPENDENT SEQUENCES[†]

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ABSTRACT

The class of eventually independent sequences for a weakly mixing transformation is an isomorphism invariant that is useful for distinguishing zero entropy transformations. This invariant is used to distinguish certain weakly mixing transformations as well as certain partially mixing transformations.

1. Introduction

Our purpose is to introduce an isomorphism invariant related to independence and apply it to study the isomorphism problem for zero entropy transformations.

Let T be an invertible weakly mixing transformation defined on the unit interval with Lebesque measure. It follows from results of Furstenberg [5] and Krengel [6] that the class of finite partitions that admit a sequence of independent iterates under T are dense in the class of all finite partitions. However, the sequence generally depends on the partition. It will be shown that for each transformation T there exist sequences $s = s(T)$ such that the partitions that eventually have independent iterates along s are dense. A partition eventually has independent iterates along s if all but a finite number of iterates are independent. We shall refer to s as an eventually independent sequence (e.i.s.) for T. An e.i.s. for T is an isomorphism invariant for T. We shall use this invariant to distinguish weakly mixing transformation and partially mixing transformations with zero entropy.

2. Preliminaries

Let $(X, \mathscr{B}, m)$ denote the unit interval with Lebesque measure and let $\mathscr{T}$ denote the class of invertible weakly mixing transformations mapping X onto X.

[†]Research partially supported by National Science Foundation Grant MCS7606735A01.
Received February 4, 1979

A finite partition of X will be denoted by $P = (p_1, p_2, \cdots, p_n)$. Let P and Q each have n atoms. A distance is defined by

$$|P - Q| = \sum_{i=1}^{n} m(p_i \Delta q_i).$$

Partitions P and Q are independent if $m(p \cap q) = m(p)m(q)$ for p in P, q in Q. In this case we denote $P \perp Q$.

Let $(k_n : n = 1, 2, 3, \cdots)$ be an increasing sequence of positive integers. The sequence $(T^{k_n}P : n = 1, 2, \cdots)$ is an independent sequence if $T^{k_i}P \perp T^{k_j}P$, $i \neq j$.

A transformation is two-sided weak mixing [6] if A, B and C in $\mathscr{B}$ imply

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |m(T^{-n}A \cap B \cap T^{n}C) - m(A)m(B)m(C)| = 0. \tag{2.1}$$

The definition of weak mixing is obtained by setting $A = X$ in (2.1). Furstenberg introduced a defintion of weak mixing of all orders and proved that weak mixing is actually equivalent to weak mixing of all orders [5]. In particular, Furstenberg's result implies that weak mixing is equivalent to two-sided weak mixing. In [6] (theorem 3.1) Krengel proved that two-sided weak mixing is equivalent to the class of partitions that admit a sequence of independent iterates under T being dense in the class of all finite partitions. We shall not use this result explicitly, but we shall refer to the method of proof of theorem 3.1 [6].

3. Results

We shall first prove Theorem 3.1 and then use it to distinguish certain weakly mixing transformations with zero entropy.

THEOREM 3.1. *A transformation T admits eventually independent sequences if and only if T is weakly mixing.*

PROOF. If T admits e.i. sequences, then T is weakly mixing by [5, 6]. For the converse, we shall first give a proof for the case when T is a Bernoulli shift. This case motivated the proof in the general case.

Let G be an independent generator for T. Given a partition P and $\varepsilon > 0$, there exists a positive integer n and $Q \subset \bigvee_{-n}^{n} T^i G$ such that $|P - Q| < \varepsilon$. The sequence of partitions $(T^{(2n+1)k}Q \colon k = 0, 1, 2, \cdots)$ is an independent sequence. This implies that any increasing sequence (k_n) with divergent gaps $(\lim_{n \to \infty}(k_{n+1} - k_n) = \infty)$ is an e.i.s. for T.

Let us now consider a general weakly mixing transformation T. Let (P_n) be a sequence of finite partitions that are dense in the class of all partitions. Hence

$$\liminf_{n\to\infty} |P_n - P| = 0 \tag{1}$$

for each partition P.

Let (ε_n) be a sequence of positive numbers decreasing to 0. The construction in [6] can be used to choose k_1 and $Q_{1,1}$ such that $|P_1 - Q_{1,1}| < \varepsilon_1$ and $T^{k_1}Q_{1,1} \perp T^{k_0}Q_{1,1}$, where $k_0 = 0$.

We now proceed by induction. At the nth stage we have positive integers $k_1 < k_2 < \cdots < k_n$ and partitions, $Q_{i,n}$, $1 \leqq i \leqq n$, such that

$$|Q_{n,n} - P_n| < \varepsilon_n, \tag{2}$$

$$|Q_{i,n-1} - Q_{i,n}| < \varepsilon_n, \qquad 1 \leqq i < n, \tag{3}$$

$$(T^{k_j}Q_{i,n})_{j=1}^{n} \text{ are independent}, \qquad 1 \leqq i \leqq n. \tag{4}$$

The construction in the proof of theorem 3.1 [6] can now be used to obtain $k_{n+1} > k_n$ and $Q_{i,n+1}$, $1 \leqq i \leqq n+1$, such that

$$|Q_{n+1,n+1} - P_{n+1}| < \varepsilon_{n+1}, \tag{5}$$

$$|Q_{i,n} - Q_{i,n+1}| < \varepsilon_{n+1}, \qquad 1 \leqq i < n+1, \tag{6}$$

$$(T^{k_j}Q_{i,n+1})_{j=i}^{n+1} \text{ are independent}, \qquad 1 \leqq i \leqq n+1. \tag{7}$$

Thus by induction we obtain a sequence of partitions $(Q_{i,n} : n \geqq i)$ for $i = 1, 2, 3, \cdots$. Now (3) implies $Q_i = \lim_{n\to\infty} Q_{i,n}$ exists if $\Sigma_{n=1}^{\infty} \varepsilon_n < \infty$. From (3) and (4) we conclude that $(T^{k_n}Q_i : n \geqq i)$ is an independent sequence. From (1) and (2) it follows that $(k_n : n \geqq 1)$ is an e.i.s. for T.

As we saw above, any sequence with divergent gaps is an e.i.s. for a Bernoulli shift. It follows from Lemma 3.3 below that a sequence with divergent gaps may not be an e.i.s. for a weakly mixing transformation. However, every e.i.s. must have divergent gaps as proved in the following lemma.

LEMMA 3.2. *If (k_n) is an* e.i.s. *for T, then* $\lim_{n\to\infty}(k_{n+1} - k_n) = \infty$.

PROOF. Let N be a positive integer. Choose a positive integer r such that $N/r < 10^{-6}$. By Rohlin's Theorem [7] we can find a set B such that T^iB, $0 \leqq i \leqq r$, are disjoint and

$$m\left(\bigcup_{i=0}^{r} T^iB\right) > 1 - 10^{-6}. \tag{1}$$

Choose s such that $A = \bigcup_{i=0}^{s} T^i B$ satisfies

$$(2) \qquad .89 \leqq m(A) \leqq .9.$$

By choice of r we have

$$(3) \qquad .88 \leqq m(A \cap T^N A) \leqq .89.$$

If E is a set such that $|E - A| < 10^{-6}$, then (3) implies

$$(4) \qquad .87 \leqq m(E \cap T^N E) \leqq .9.$$

Thus $T^{k_i}E \perp T^{k_{i+1}}E$ implies $k_{i+1} - k_i > N$. This yields the desired conclusion.

The following result implies that if T is weakly mixing but not mixing, then there exist divergent sequences that are not e.i.s. for T.

Lemma 3.3. *If there exist* (k_n) *and* B *such that* $\lim_{n\to\infty} m(T^{k_n}B \cap B) \neq m(B)^2$, *then* (k_n) *is not an* e.i.s. *for* T.

Proof. There exist $\varepsilon > 0$ and a subsequence (j_n) of (k_n) such that either (1) or (2) hold:

$$(1) \qquad \lim_{n\to\infty} m(T^{j_n}B \cap B) > m(B)^2 + \varepsilon,$$

$$(2) \qquad \lim_{n\to\infty} m(T^{j_n}B \cap B) < m(B)^2 - \varepsilon.$$

We assume (1) holds. If $|E - B| < 10^{-6}\varepsilon$, then (1) implies

$$(3) \qquad m(T^{j_n}E \cap E) > m(E)^2 + \varepsilon/2.$$

From (3) we conclude $T^{j_n}E \perp E$ is impossible; hence (k_n) cannot be an e.i.s. for T.

From Lemmas 3.2 and 3.3 we obtain

Corollary 3.4. *If T is weakly mixing, then every* e.i.s. *for T has divergent gaps. If T is not mixing, then there exist sequences with divergent gaps that are not* e.i.s. *for T.*

The proof of Lemma 3.3 can be extended to obtain the following result.

Lemma 3.5. *If T is not mixing of order p for some p, then there exist sequences with divergent gaps that are not* e.i.s. *for T.*

4. Weak mixing

We shall construct a class $\mathscr{C}$ of weak mixing transformations that are not mixing and have zero entropy by extending the construction of Chacon [1]. It will be shown that given any sequence with divergent gaps, there exists a transformation in the class for which this sequence is not an e.i.s. In particular, this implies that $\mathscr{C}$ contains nonisomorphic transformations.

A transformation T in $\mathscr{C}$ is constructed as follows. We start with a single column C_1 of intervals. This column is cut into three equal subcolumns. An additional interval is placed above the middle subcolumn. The three subcolumns are then stacked consecutively to form one column C_2. At an odd stage in the construction we have a column C_{2n-1}. We obtain a column C_{2n} in the same way that C_2 was obtained from C_1. At an even stage we have a column C_{2n} of height h_{2n}. This column is cut into u_{2n} equal subcolumns. No more than h_{2n} additional intervals are placed above the last subcolumn. The u_{2n} columns are then stacked consecutively to form one column C_{2n+1}. In the construction it is assumed that u_{2n} is sufficiently large so that $\Sigma_{n=1}^{\infty} h_{2n}/u_{2n} < \infty$. This guarantees that the total measure space on which T is defined is finite.

THEOREM 4.1. *Let (k_l) be a sequence with divergent gaps. There exists T in $\mathscr{C}$ such that (k_l) is not an* e.i.s. *for T.*

PROOF. Let (ε_{2n}) be a sequence of positive numbers that decrease to 0. Let C_1 have height 2; hence C_2 has height 7. Choose l_2 so that $t_2 = k_{l_2}$ satisfies $7/t_2 < \varepsilon_2$. Let $u_2 = [7/t_2]$ and let $v_2 = t_2 - 7u_2$; hence $t_2 = v_2 + 7u_2$. Cut C_2 into u_2 equal subcolumns and add v_2 intervals above the last subcolumn. The u_2 subcolumns are now stacked consecutively to form C_3. Note that C_3 has height t_2. We now proceed inductively to define C_{2n+1}.

Given column C_{2n} of height h_{2n}, we choose $t_{2n} = k_{l_{2n}}$ so large that $h_{2n}/t_{2n} < \varepsilon_{2n}$. Let $u_{2n} = [t_{2n}/h_{2n}]$ and let $v_{2n} = t_{2n} - h_{2n}u_{2n}$. Cut C_{2n} into u_{2n} equal subcolumns and add v_{2n} intervals above the last subcolumn. The u_{2n} subcolumns are now stacked consecutively to form C_{2n+1}. Note that C_{2n+1} has height t_{2n}.

If ε_n decreases to 0 sufficiently fast, then the total added measure will be finite. The construction of C_{2n} from C_{2n-1}, $n \geqq 1$, implies that T is weak mixing [1].

Let I be an interval in C_{2n+1}. Since C_{2n+1} has height t_{2n}, the construction of C_{2n+2} implies

$$m(T^{t_{2n}}I \cap I) \geqq m(I)/3. \tag{1}$$

Since the intervals in C_{2n+1}, $n \geqq 1$, generates $\mathscr{B}$, (1) implies

$$\limsup_{n\to\infty} m(T^{t_{2n}}B \cap B) \geqq m(B)/3, \qquad B \in \mathscr{B}. \tag{2}$$

Now (2) implies T is not mixing. Also (2) implies that (t_{2n}) cannot be an e.i.s. for T; hence (k_l) cannot be an e.i.s. for T.

The fact that T has zero entropy is implied by the intervals in C_n, $n \geqq 1$, generating $\mathscr{B}$.

Let $T_1 \in \mathscr{C}$; hence Theorem 3.1 implies there exists an e.i.s. $s(T_1)$ for T_1. Now $s(T_1)$ has divergent gaps by Lemma 3.2. Theorem 4.1 implies there exists $T_2 \in \mathscr{C}$ and $s(T_1)$ is not an e.i.s. for T_2. Thus T_1 and T_2 cannot be isomorphic. It can also be shown that there exist a countable class of non-isomorphic transformations in $\mathscr{C}$. An open problem is to show there exist an uncountable class of non-isomorphic transformations in $\mathscr{C}$.

5. Partially mixing

A transformation T is partially mixing if there exists $\beta > 0$ such that A and B in $\mathscr{B}$ imply

$$\liminf_{n\to\infty} m(T^nA \cap B) \geqq \beta m(A)m(B). \tag{5.1}$$

A partially mixing transformation is α-mixing, $0 < \alpha < 1$, if (5.1) holds for $\beta = \alpha$ but does not hold for $\beta = \alpha + \varepsilon$, $\varepsilon > 0$. Partially mixing transformations were introduced in [3] and α-mixing transformations were constructed in [4] for each α, $0 < \alpha < 1$. The transformations in [3, 4] also have zero entropy.

A partially mixing transformation is weakly mixing [2]. The weakly mixing transformations in the class $\mathscr{C}$ constructed above are not partially mixing. Note that an α-mixing transformation cannot be isomorphic to a β-mixing transformation if $\alpha \neq \beta$.

For each α, $0 < \alpha < 1$, the construction in [4] can be extended to obtain a class $\mathscr{C}_\alpha$ of α-mixing transformations with zero entropy. It is possible to construct the class $\mathscr{C}_\alpha$ so that the following result holds.

THEOREM 5.2. *Let (k_l) be a sequence with divergent gaps. There exists T in $\mathscr{C}_\alpha$ such that (k_l) is not on* e.i.s. *for T.*

A proof of Theorem 5.2 can be obtained by a more detailed construction in the formation of the column V_5 in the construction in [4]. Essentially one restricts the construction at the even stages in §4 above to V_5 and this guarantees (k_i) is not an e.i.s. for T. We shall omit the details.

It follows from Theorem 5.2 that there exist non-isomorphic α-mixing zero entropy transformations in $\mathscr{C}_\alpha$.

In general, if a transformation T has zero entropy, then there will exist sequences with divergent gaps which cannot be e.i. sequences for T. In particular, there exist mixing transformations with zero entropy; hence there exist sequences with divergent gaps that are not e.i. sequences for certain mixing transformations. In a subsequent paper we shall consider the problem of constructing a mixing transformation that does not admit a given sequence with divergent gaps as an e.i.s.

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